

Quantum Logic and Non-Commutative Geometry

P. A. Marchetti^{1,2} and R. Rubele^{3,4}

Received April 28, 2004; accepted May 25, 2004
Published Online: November 30, 2006

We propose a general scheme for the “logic” of elementary propositions of physical systems, encompassing both classical and quantum cases, in the framework given by Non-Commutative Geometry. It involves Baire*-algebras, the non-commutative version of measurable functions, arising as envelope of the C^* -algebras identifying the topology of the (non-commutative) phase space. We outline some consequences of this proposal in different physical systems. This approach in particular avoids some problematic features appearing in the definition of physical states in the standard (W^* -)algebraic approach to classical mechanics.

KEY WORDS: quantum logic; non-commutative geometry; Baire*-algebras

1. INTRODUCTION

In many respects Non-Commutative Geometry (NCG) (Connes, 1994) appears as the most complete mathematical setting for a unified description of quantum and classical physical systems, besides being a source of some highly imaginative ideas in the attempt of constructing a unified theory of fundamental forces including gravity (see e.g. Doplicher *et al.*, 1995; Froehlich *et al.*, 1995; Connes *et al.*, 1998; Witten, 2001, and references therein).

In this paper we propose a characterisation of the lattice of elementary propositions, i.e. the “logic”, of quantum and classical systems which appears to fit naturally in the framework of NCG and solves some problematic feature of the more standard W^* -algebraic approach (see e.g. Thirring, 1981; Primas, 1983; Haag, 1992; Redei, 1998).

A root of Non-Commutative Geometry is the idea that one can generalise many branches of standard functional analysis, such as measure theory, topology

¹ Dipartimento di Fisica, Università degli Studi di Padova, Italia; e-mail: marchetti@pd.infn.it

² Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Via F. Marzolo, 8, 35131 Padova, Italia.

³ Dipartimento di Fisica “E.R. Caianiello,” Università degli Studi di Salerno.

⁴ Istituto Nazionale di Fisica della Materia, Unità di Salerno, Via S. Allende, 84081 Baronissi (SA), Italia; e-mail: rubele@sa.infn.it

and differential geometry, by replacing the commutative algebras of functions over some space X by a suitable non-commutative algebra which may in a sense be interpreted as the “algebra of functions over a non-commutative space.”

In the commutative case one can consider various degrees of regularity of the functions ranging from measurable, to continuous, to smooth. The non-commutative analogue of the algebra of complex bounded continuous functions is a C^* -algebra, whereas spaces of complex essentially bounded measurable functions (\mathcal{L}^∞) are generalized by von Neumann algebras or, in abstract form, W^* -algebras. Algebraic generalizations of spaces of smooth functions are pre- C^* -algebras, i.e. $*$ -subalgebras of a C^* -algebra closed under the holomorphic functional calculus. Probability measures on spaces of continuous functions find a non-commutative generalization in the concept of algebraic states, henceforth simply states: the linear positive normalized functionals on a C^* -algebra; in particular Dirac measures with support on one point are generalized by pure states, i.e. states that cannot be written as convex combinations of other states. (Notice that since a C^* -algebra is a Banach space, states are elements of its dual as they are continuous being bounded.)

A link with quantum theory appears when quantum mechanics is interpreted as a “mechanics over a non-commutative phase space” in the spirit of Heisenberg and Dirac. If we consider a quantum non-relativistic elementary particle with classical analogue i.e. without internal degrees of freedom, the appropriate algebra of “smooth functions, or observables, in phase space” is the Weyl algebra generated by the bounded version,

$$e^{i\vec{\alpha}\cdot\vec{q}} e^{i\vec{\beta}\cdot\vec{p}} = e^{i\vec{\beta}\cdot\vec{p}} e^{i\vec{\alpha}\cdot\vec{q}} e^{i\hbar\frac{\vec{\alpha}\cdot\vec{\beta}}{2}}, \quad (1)$$

of the celebrated Heisenberg commutation relations:

$$q_i p_j - p_j q_i = i\hbar\delta_{ij}, \quad (2)$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and $\{q_i\}_{i=1}^3$ and $\{p_i\}_{i=1}^3$ are the “coordinates” of the “non-commutative phase space” corresponding respectively to canonical coordinates of the underlying commutative classical configuration space and their conjugate momenta. It turns out that the corresponding C^* -algebra of “continuous bounded observables” is isomorphic to the C^* -algebra $\mathfrak{K}(\mathcal{H})$ of compact operators on an infinite dimensional separable complex Hilbert space, \mathcal{H} , and the W^* -algebra of “bounded measurable observables” is isomorphic to the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on \mathcal{H} . (The qualification “continuous bounded” used above is meant to evocate the analogy with the commutative case and is not referred to norm continuity of operators on Hilbert spaces, which of course is equivalent to boundedness.)

A relation with quantum logic then appears as follows. It has been recognised in the seminal work of Birkhoff and von Neumann (1936), that the system of elementary physical propositions corresponding to yes-no experiments of quantum mechanics can be represented as the complete orthomodular lattice of closed subspaces of a separable Hilbert space \mathcal{H} . (Actually, orthomodularity was not introduced in Birkhoff and von Neumann, 1936, but by Piron, 1964; for an historical comment see Redei, 1998.) Such lattice can be characterized also algebraically in terms of the associated orthogonal projectors, p , in \mathcal{H} , with the well known definitions of orthocomplement $^\perp$, meet \wedge and join \vee operations: $p^\perp = \mathbf{1} - p$, $p_1 \wedge p_2 = \lim_{n \rightarrow \infty} (p_1 p_2)^n = \lim_{n \rightarrow \infty} (p_2 p_1)^n$, $p_1 \vee p_2 = (p_1^\perp \wedge p_2^\perp)^\perp$ and partial ordering defined by $p_1 \leq p_2$ iff $p_1 = p_1 \wedge p_2$. In turn, the projectors are the self-adjoint elements of the von Neumann algebra $\mathfrak{B}(\mathcal{H})$ satisfying $p^2 = p$. The set of projectors of any W^* -algebra has the structure of a complete orthomodular lattice with lattice operations defined algebraically as above. Therefore it has been proposed to identify as a model for the propositional lattice of physical systems the lattice of projectors of a W^* -algebra.

A classical system in this setting is given in terms of a commutative W^* -algebra; the corresponding lattice of propositions is therefore distributive, i.e. a Boolean algebra. Hence the transition from the classical to the quantum level corresponds to the elimination of the commutativity postulate, due to the existence of the universal constant \hbar , which is replaced by 0 in classical mechanics. More precisely, for a classical particle the W^* -algebra generated by the commutation relation (1) with $\hbar = 0$ is taken to be $\mathcal{L}^\infty(\Omega, \omega_L)$ where Ω is the phase space and ω_L is the Lebesgue measure on Ω which coincides with the Liouville measure given in terms of the symplectic form.

Although the W^* -approach has the great virtue of describing classical and quantum systems and the related logics in a unifying canonical scheme, it reveals some drawbacks in the definition of states at the classical level. In the algebraic approach the states describe the “states of knowledge” of the observable quantities and pure states correspond to maximal knowledge. However in classical systems points in phase space are of zero ω_L -measure and hence “invisible” to $\mathcal{L}^\infty(\Omega, \omega_L)$. Therefore, as already noticed by von Neumann, it is not naturally defined in this setting the most fundamental state of classical mechanics corresponding to a single point in phase space selecting “initial conditions” of the system; see Halvorson (2001) for a more refined and recent analysis of the problem. Although this fact could be attributed to the practical impossibility of a precise measurement, it is at least philosophically somewhat unnatural. For a related problem e.g. Teller (1995) argued that “*if we believe that systems possess exact values for continuous quantities, classical theory contains the descriptive resources for attributing such values to the system, whether or not measurements are taken to be imprecise in some sense.*”

Instead, points in phase space can be taken as support of Dirac measures and these are naturally defined as states on $C_0(\Omega)$, the C^* -algebra of bounded continuous functions on Ω vanishing at infinity, generated by the commutation relations (1) with $\hbar = 0$. However $C_0(\Omega)$ does not contain non-trivial projectors, since these are characteristic functions which are not continuous. Analogously the C^* -algebra of compact operators on a separable Hilbert space, $\mathfrak{K}(\mathcal{H})$, generated by (1) with $\hbar \neq 0$, does not contain a lattice of projectors even σ -complete, i.e. stable under a countable number of meet and join operations, and this is the weakest reasonable completeness to require in a logic, excluding “unsharp” approaches, see e.g. Dalla Chiara and Giuntini (2001) (we use the word “complete” to denote stability under an arbitrary, even non countable, number of meet and join operations). On the other hand the pure states on $\mathfrak{K}(\mathcal{H})$ are exactly in correspondence with the rays of \mathcal{H} , as required on physical grounds. In fact the dual of $\mathfrak{K}(\mathcal{H})$ is isomorphic to the space of trace-class operators on \mathcal{H} , the condition of positivity and normalization then identifies the states as the “statistical matrices.” The pure states correspond to one dimensional projectors hence to rays, but this correspondence does not hold for the pure states on $\mathfrak{B}(\mathcal{H})$, which include also unphysical “improper states.” (To save the physically required correspondence in this case one has to restrict to the normal states, i.e those which are completely additive.)

Hence in a NCG setting as a natural framework to embed an algebraic model of elementary propositions one is naturally looking for a “space” in general larger than the C^* -algebra of “continuous bounded observables” \mathfrak{A} , but smaller than the W^* -algebra of “essentially bounded measurable observables,” and containing a σ -complete orthomodular lattice of projectors. Furthermore one would like this space still to be some “closure” of the C^* -algebra \mathfrak{A} , which in the NCG approach identifies the topology of the non-commutative phase space and is taken as the basic algebra, identifying the space of physical states.

This “space” in fact exists, it is called Baire*-algebra and can be described in the above terminology as the C^* -algebra of (Baire) measurable bounded functions or observables on a generally “non-commutative” space, and it is generated by \mathfrak{A} , as a suitable enveloping algebra. We denote it by $\mathcal{B}(\mathfrak{A})$.

We propose to identify the lattice of projectors of $\mathcal{B}(\mathfrak{A})$, denoted by $\mathbb{P}(\mathcal{B}(\mathfrak{A}))$, as a model for the lattice of elementary propositions of the physical systems described by \mathfrak{A} , and to identify the logical states ϕ_L [see next section] as the restriction to $\mathbb{P}(\mathcal{B}(\mathfrak{A}))$ of the lift $\tilde{\phi}$ to $\mathcal{B}(\mathfrak{A})$ of states ϕ on \mathfrak{A} . If $a \in \mathcal{B}(\mathfrak{A})$, then $\tilde{\phi}(a)$ is the expectation value of the measurable observable a in the state $\tilde{\phi}$ and in particular if p is a projector in $\mathcal{B}(\mathfrak{A})$, then $\tilde{\phi}(p) = \phi_L(p) \in [0, 1]$ yields the probability that the proposition represented by p is true in the logical state ϕ_L .

As it will be discussed in Section 3, this setting solves the above quoted difficulty of the W^* -algebra approach. The scheme can be summarized by means

of the following commutative diagram

$$\begin{array}{ccccc}
 \mathbb{P}(\mathcal{B}(\mathfrak{A})) & \xrightarrow{i} & \mathcal{B}(\mathfrak{A}) & \xleftarrow{j} & \mathfrak{A} \\
 & \searrow \phi_L & & \swarrow \tilde{\phi} & \swarrow \phi \\
 & & [0,1] & \xrightarrow{k} & \mathbb{C}
 \end{array}$$

where i, j and k are the obvious injections. We remark that a consequence of this proposal is that the lattice of elementary propositions of a physical quantum system, although always orthomodular σ -complete it is not always complete, nor atomic, nor Hilbertian (i.e. isomorphic to all the orthogonal subspace of a separable Hilbert space). These specific features are encoded in the C^* -algebra of “continuous bounded observables” \mathfrak{A} of the system. More obviously, for classical systems \mathfrak{A} is abelian and this implies a distributive property for the lattice of propositions.

In the rest of this paper we will make mathematically precise the setting described above. Although the mathematical results presented here are not original the overall scheme and its degree of generality to the best of our knowledge are novel.

2. LOGICAL STATES

Let \mathcal{L} be the orthomodular σ -complete lattice assumed to describe the set of elementary propositions of a physical system. A logical state (in the sense of Mackey–Jauch–Piron (Mackey, 1963; Jauch, 1968; Piron, 1976)) ϕ_L is a σ -orthoadditive map from \mathcal{L} to $[0,1]$; more explicitly, if P is a proposition in \mathcal{L} , then $\phi_L(P^\perp) = 1 - \phi_L(P)$, and if $\{P_i\}_{i \in I}$ is a countable number of propositions pairwise orthogonal, i.e. $P_i \leq P_j^\perp$ for $i \neq j$, then $\phi_L(\bigvee_i P_i) = \sum_i \phi_L(P_i)$. $\phi_L(P)$ is the probability that the proposition P is true in the state ϕ_L . A logical state ϕ_L is “pure” if it cannot be written as a convex combination of other logical states i.e. if for any two logical states ϕ_1 and ϕ_2 the equation $\phi_L = \alpha\phi_1 + (1 - \alpha)\phi_2, 0 < \alpha < 1$, implies $\phi_L = \phi_1 = \phi_2$. Pure logical states correspond to the maximal knowledge attainable on the propositional system. A logical state is called “normal” if is completely orthoadditive.

In the W^* -algebraic approach we have the following:

Theorem 2.1. (Cirelli and Gallone, 1973) *Identifying as a model for \mathcal{L} the lattice of projectors of a W^* -algebra \mathfrak{M} , the restriction to $\mathbb{P}(\mathfrak{M})$ of normal states*

on \mathfrak{M} are normal logical states; furthermore pure logical states corresponds to restriction of pure states.

As discussed in the introduction the proposal to identify the logical states as restriction of normal states on W^* -algebras excludes the states corresponding to single points in phase space in classical systems unless the phase space is discrete in view of the following:

Theorem 2.2. *A state on a W^* -algebra \mathfrak{M} is normal iff it is an element of its predual \mathfrak{M}_* .*

Since for classical systems $\mathfrak{M} = \mathcal{L}^\infty(\Omega, \omega_L)$ and $\mathfrak{M}_* = \mathcal{L}^1(\Omega, \omega_L)$, this excludes the Dirac measures concentrated on one point of Ω as they do not belong to $\mathcal{L}^1(\Omega, \omega_L)$.

On the other hand the requirement of normality is perfectly suited for a standard (i.e. without, or at most with a countable set of superselection sectors) quantum mechanical system with finite dynamical degrees of freedom, where we know that physical states are “statistical matrices,” which are positive trace 1 elements of $J_1(\mathcal{H})$, the space of trace-class operators on the separable Hilbert space of physical vector states \mathcal{H} . In this case, in fact, $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ and $\mathfrak{M}_* = J_1(\mathcal{H})$.

The choice of a W^* or von Neumann algebra as foundational in a C^* approach is also mathematically not entirely natural in NCG, as Connes (1993) pointed out: “It is true, and at first confusing, that any von Neumann algebra is a C^* -algebra, but not an interesting one because it is usually not norm separable. For instance let (X, μ) be a diffuse probability space (every point $p \in X$ is μ -negligible), then $\mathcal{L}^\infty(X, \mu)$ is a von Neumann algebra but it is not norm separable and its spectrum [see definition after Theorem 3.1 and comment after Definition 3.2] as a C^* -algebra is a pathological space that has little to do with the original standard Borel space X .”

The somewhat unsatisfactory situation outlined above is avoided if we introduce the notion of Baire*-algebra.

3. BAIRE*-ALGEBRAS

To put in a proper perspective the definition of a Baire*-algebra it is convenient to recall some basic notions of the theory of Baire functions, whose space is the commutative version of a Baire*-algebra.

Let (X, Σ) be a measure space, where Σ denotes a σ -algebra of subsets of X . A real or complex function is Σ -measurable if $f^{-1}(B) \in \Sigma$ for every B borelian in \mathbb{R} or in \mathbb{C} , respectively. A class \mathcal{F} of real functions over X is called monotonically sequentially complete if every limit of a monotonic sequence of functions of \mathcal{F} belongs to \mathcal{F} . The class of real Σ -measurable functions is an

algebra monotonically complete σ -stable under the lattice operations of meet and join.

Let X be a locally compact topological space. A compact set of X is of type G_δ if it is a countable intersection of open sets of X . The class of G_δ compacts generates the σ -algebra B_X of the Baire sets of X . This is the smallest σ -algebra from which one can reconstruct the topology of X (Halmos, 1950).

A real function on X is called a Baire function if it is B_X -measurable; a complex function is a Baire function if both its real and imaginary part are Baire functions. The class of real Baire functions is the smallest class including all continuous function in X and the limit of every bounded monotone sequence of them. The class of complex Baire functions on X will be denoted by $\mathcal{B}(X)$. If X is a metric space then the σ -algebra of Baire sets coincides with the σ -algebra of Borel sets, generated by the open sets of X , and the Baire functions are Borel functions. For this reason Baire*-algebras were called Borel*-algebras in Pedersen (1979). To each point $p \in X$ is associated a Dirac measure $d\mu_p$ on $\mathcal{B}(X)$ with support $\{p\}$ and mass 1.

To discuss the generalization to a non-commutative setting we need some preliminary definitions which extend to such a setting the basic notions involved in the constructions outlined above. A C^* -algebra \mathfrak{A} is called monotonically sequentially complete if every bounded monotone sequence of the self-adjoint part of \mathfrak{A} , \mathfrak{A}_{sa} , possesses a limit in \mathfrak{A}_{sa} . A state ϕ over a monotonically sequentially complete C^* -algebra \mathfrak{A} is called σ -normal if for every bounded monotone sequence $\{x_n\}_{n \in \mathbb{N}}$ in \mathfrak{A}_{sa} we have

$$\phi\left(\bigvee_n x_n\right) = \bigvee_n \phi(x_n).$$

Definition 3.1. (Pedersen, 1979) A C^* -algebra \mathcal{B} is called a Baire*-algebra if it is monotonically sequentially complete and it admits a separating family of σ -normal states.

Notice that, as discussed below, in the commutative case $\mathcal{B}(X)$ is a Baire*-algebra with separating family of σ -normal states generated by the Dirac measures $\{d\mu_p\}_{p \in X}$.

An important result connecting Baire* and W^* -algebras is the following:

Theorem 3.1. *If a Baire*-algebra has a faithful representation in a separable Hilbert space, then it is isomorphic to a W^* -algebra.*

There is a natural “closure” of a C^* -algebra to obtain a Baire*-algebra. To present this construction we need some preliminary definitions. Given a C^* -algebra \mathfrak{A} , let $\hat{\mathfrak{A}}$, be its spectrum, i.e. the set of (equivalence classes of unitarily equivalent) irreducible representations of \mathfrak{A} . Let ϕ be a (representative) pure state

corresponding to a point of $\hat{\mathfrak{A}}$, and π_p the corresponding representation. The atomic representation of \mathfrak{A} is given by $\pi_a = \bigoplus_{\phi \in \hat{\mathfrak{A}}} \pi_\phi$ and it is a faithful representation of \mathfrak{A} .

Then we have the following:

Definition 3.2. (Baire* enveloping algebra) (Pedersen, 1979) Given a C^* -algebra, \mathfrak{A} , and a subset $M \subset \mathfrak{A}_{sa}$, we define the monotone sequential closure of M , $\mathcal{B}(M)$, as the smallest subset of the atomic representation $\pi_a(\mathfrak{A}_{sa})$, containing $\pi_a(M)$ and the limit of every monotone sequence of elements of $\pi_a(M)$. The Baire* enveloping algebra of \mathfrak{A} , is given by

$$\mathcal{B}(\mathfrak{A}) \equiv \mathcal{B}(\mathfrak{A}_{sa}) + i\mathcal{B}(\mathfrak{A}_{sa}).$$

$\mathcal{B}(\mathfrak{A})$ is a Baire*-algebra with the family of σ -normal states given by the unique extension of the states on \mathfrak{A} to $\mathcal{B}(\mathfrak{A})$.

To better understand the meaning of the Baire* enveloping algebra notice that if \mathfrak{A} is commutative and separable, then by the Gel'fand isomorphism (see e.g. Pedersen, 1979; Thirring, 1981), the spectrum $\hat{\mathfrak{A}}$ is a locally compact Hausdorff space and \mathfrak{A} is isomorphic to $C_0(\hat{\mathfrak{A}})$, the space of continuous function in $\hat{\mathfrak{A}}$ vanishing at infinity (if \mathfrak{A} is non-compact). Therefore $\mathcal{B}(\mathfrak{A}) = \mathcal{B}(\hat{\mathfrak{A}})$, i.e. the enveloping Baire*-algebra is exactly the algebra of complex Baire functions on \mathfrak{A} . Conversely if $\mathfrak{A} = C(X)$ with X locally compact, $\hat{\mathfrak{A}} \simeq X$ as a topological space and $\widehat{\mathcal{B}}(\mathfrak{A}) \simeq X$ as a Borel space. The irreducible representations correspond to pure states given by the normalised Dirac measures $\{d\mu_p\}_{p \in \hat{\mathfrak{A}}}$.

Notice that since $\mathcal{B}(\hat{\mathfrak{A}})$ has no faithful representations on a separable Hilbert space unless \mathfrak{A} is discrete, then in general the commutative Baire*-algebra $\mathcal{B}(\mathfrak{A})$ is not a W^* -algebra. However we have the following result refining the previous one:

Theorem 3.2. (Davies, 1968) *If \mathfrak{A} has a faithful representation π on a separable Hilbert space then $\mathcal{B}(\pi(\mathfrak{A})) \simeq \pi(\mathfrak{A})'$ i.e. it is isomorphic to the von Neumann algebra generated by $\pi(\mathfrak{A})$ and its σ -normal states are the normal states of the von Neumann algebra.*

For the logical interpretation, the crucial property of Baire*-algebras is the following:

Theorem 3.3. *The set of projectors $\mathbb{P}(\mathcal{B})$ of a Baire*-algebra \mathcal{B} is an orthomodular σ -complete lattice.*

Furthermore, since the extensions to $\mathcal{B}(\mathfrak{A})$ of the states on \mathfrak{A} are σ -normal, we have:

Proposition. *The restriction of the σ -normal states of the Baire*-enveloping algebra $\mathcal{B}(\mathfrak{A})$ to $\mathbb{P}(\mathcal{B}(\mathfrak{A}))$ are logical states.*

The identification of Baire*-algebras as the abstract setting for bounded measurable observables is the one that makes it transparent the interpretation of quantum mechanics as a “theory of quantum probability.” Although there is a high amount of papers written on this topic, it seems that a framework like the one we are outlining here is not considered. As an example, in a quite recent general review on the subject (Streater, 2000), R. F. Streater pointed out that: “*Though the classical axioms were yet to be written down by Kolmogorov, Heisenberg, with help of the Copenhagen interpretation, invented a generalisation of the concept of probability, and physicists showed that this was the model of probability chosen by atoms and molecules.*” However, the algebraic (W^* -)approach envisaged therein appears less close than ours to the standard treatment of probability on topological measure spaces, where the Borel or Baire structure is determined by the topology, as $\mathcal{B}(\mathfrak{A})$ is determined by \mathfrak{A} .

We end this section with a

Remark. In the definition of enveloping Baire*-algebra we can replace the atomic representation π_a with the universal representation $\pi_u = \bigoplus_{\phi \in S(\mathfrak{A})} \pi_\phi$, where $S(\mathfrak{A})$ is the set of states on \mathfrak{A} and the corresponding $\mathcal{B}(\mathfrak{A})$ is isomorphic to the one defined via π_a . Then $\mathcal{B}(\mathfrak{A}) \subset \pi_u(\mathfrak{A})''$, which is the universal enveloping von Neumann algebra of \mathfrak{A} .

Therefore the σ -complete orthomodular lattice of $\mathcal{B}(\mathfrak{A})$ describing the elementary propositions of the system characterized by \mathfrak{A} can be embedded in the complete orthonormal lattice of $\pi_u(\mathfrak{A})''$; for the relevance of the existence of the embedding from the logical point of view see Dalla Chiara and Giuntini (2001).

4. CONSEQUENCES FOR THE LOGIC OF PHYSICAL SYSTEMS

Using the notions introduced in the previous section one can make precise the scheme outlined in the Introduction. At the foundational level one considers the algebra of “continuous bounded observables” of the physical system, described by a C^* -algebra \mathfrak{A} , possibly given as the closure of a pre- C^* -algebra of “smooth observables,” and the states on \mathfrak{A} giving the expectation values of the observables. The algebraic realization of the lattice of elementary propositions corresponding to yes-no experiments, concerning the system described by \mathfrak{A} is given by the σ -complete orthomodular lattice of the projectors of the Baire* enveloping algebra $\mathcal{B}(\mathfrak{A})$, i.e. $\mathbb{P}(\mathcal{B}(\mathfrak{A}))$. Logical states are given by the restriction to $\mathbb{P}(\mathcal{B})$ of the lift to $\mathcal{B}(\mathfrak{A})$ of the algebraic states on \mathfrak{A} . Then pure logical states describing maximal knowledge correspond to pure states on \mathfrak{A} ; notice that in general they are not pure states of $\mathcal{B}(\mathfrak{A})$.

Let us comment on some implications of the above scheme for the logic of elementary propositions of physical systems.

(1) Systems in classical mechanics.

If the phase space Ω of the system is a locally compact Hausdorff space, then $\mathfrak{A} = C_0(\Omega)$ and $\mathcal{B}(\mathfrak{A}) = \mathcal{B}(\Omega)$. The states on \mathfrak{A} are the regular Borel probability measures which have a unique extension to $\mathcal{B}(\Omega)$. Pure states are Dirac measures $\{d\mu_p\}_{p \in \Omega}$ with support on one point in phase space, hence solving the problem outlined in the Introduction.

Remark. This solution was first envisaged in Davies (1968) and Plymen (1968) where instead of Baire* enveloping algebras, Σ^* enveloping algebras were used, roughly speaking replacing monotone sequential closure with weak sequential closure. In particular in the abelian case the two concepts coincide.

The lattice of propositions $\mathbb{P}(\mathcal{B}(\Omega))$ is both atomic and distributive. As always in the algebraic setting, there is a direct correspondence between the abelian structure of the algebra of observables characterising their classical nature and the distributive property of the lattice of elementary propositions.

(2) Quantum mechanical system with countable superselection sectors.

Example. *Quantum mechanics of an elementary particle without spin.*

The algebra \mathfrak{A} is the C^* -algebra generated by the Weyl commutation relations (1) and it is isomorphic to $\mathfrak{K}(\mathcal{H})$ with \mathcal{H} separable infinite dimensional; in view of Theorem 3.2, $\mathcal{B}(\mathfrak{A}) \simeq \mathfrak{K}(\mathcal{H})'' \simeq \mathfrak{B}(\mathcal{H})$; the σ -normal states correspond to the statistical matrices. $\mathbb{P}(\mathcal{B}(\mathfrak{A}))$ is atomic and Hilbertian. In this specific example it is also irreducible, in correspondence with the absence of superselection sectors. Notice that in the Baire approach for classical system naturally appear the Dirac measures excluded in the W^* approach, whereas in quantum mechanics are naturally excluded the singular, i.e. non-normal, states of the above approach. By the way, our approach also provides a natural justification for the choice made e.g. in Duvenhage (2002) (see also Clifton *et al.* (2003) for a variant) to discuss information theory in the algebraic setting using measurable functions in classical mechanics and bounded operators in quantum mechanics.

Remark. The Baire approach permits also to avoid a problematic feature appearing in the definition of states in the temporal logic approach proposed in Primas (1983), where, motivated by ontological considerations (which of course one may not agree with), a distinction is made between “ontic” states and “epistemic” logical states. Let \mathcal{L} be the orthomodular σ -complete lattice assumed to describe the set of proposition of a physical system. An “ontic” state is a lattice ortho-homomorphism ρ of a maximal orthomodular sublattice \mathcal{T} of \mathcal{L} into \mathcal{B}_2 , the Boolean algebra of truth values. The requirement on \mathcal{T} to be maximal means that it does not exist an orthomodular sublattice \mathcal{T}' , containing properly \mathcal{T} , to which ρ can be extended as

ortho-homomorphism in \mathcal{B}_2 . This requirement corresponds to the physical intuition of a state with “maximal information” and in the algebraic approach these are the pure algebraic states. An “ontic” state is called normal if ρ is a σ -homomorphism. In this approach an “ontic” state refers to “actualized” properties the system has (at some time). States which refer to our knowledge are called “epistemic.” On this basis, if \mathcal{L} is the lattice of projectors of a W^* -algebra \mathfrak{M} , ontic states are identified with (arbitrary, even non normal) pure states and epistemic states with normal states. Therefore “ontic” states are not a subset of “epistemic” states. Furthermore only for normal states it has been proved that every ontic states on $\mathbb{P}(\mathfrak{M})$ has a unique extension to a pure state of \mathfrak{M} and every pure state on \mathfrak{M} defines a unique ontic state. For non normal states the situation appear obscure, in particular for W^* -algebras that do not admit pure normal states! Instead in the Baire approach, i.e. if $\mathcal{L} = \mathbb{P}(\mathcal{B}(\mathfrak{A}))$, one could simply identify “epistemic” states with the σ -normal states and the “ontic” would be those corresponding to the lift of the pure states on \mathfrak{A} , thus a subset of the epistemic.

(3) Quantum mechanical system with non countable superselection sectors.

Example. Quantum mechanics of an elementary particle without spin on a circle S^1 .

The algebra \mathfrak{A} is the C^* -algebra generated by the Weyl commutation relations

$$e^{in\varphi} e^{i\beta p} = e^{i\beta p} e^{in\varphi} e^{i\hbar \frac{n\beta}{2}}$$

where φ is the angle parametrizing the circle S^1 , $n \in \mathbb{Z}$, $\beta \in [0, 2\pi]/\hbar$.

Inequivalent irreducible representations are labelled by an angle $\theta \in [0, 2\pi)$ and the corresponding Hilbert space will be denoted by \mathcal{H}_θ , see e.g. Thirring (1981) and for a deeper analysis Cavallaro *et al.* (1999). These are the so-called θ -sectors and they arise physically e.g. in models where the particle is charged and coupled to a vector potential whose magnetic field strength is supported in a region in the interior of the disk bounded by the circle S^1 , in the region forbidden for the particle motion.

A magnetic flux Φ through the disk induces a representation of \mathfrak{A} labelled by $\theta = \Phi \bmod 2\pi$. Hence $\mathfrak{A} \simeq \oplus_\theta \mathfrak{K}(\mathcal{H}_\theta) \simeq C(S^1, \mathfrak{K}(\mathcal{H}))$ with \mathcal{H}_θ and \mathcal{H} separable infinite dimensional. $\mathcal{B}(\mathfrak{A}) \simeq \mathcal{B}(S^1, \mathfrak{B}(\mathcal{H}))$, the Baire (or Borel) functions on S^1 , $\mathfrak{B}(\mathcal{H})$ -valued.

$\mathbb{P}(\mathcal{B}(\mathfrak{A}))$ is atomic, coincides with the lattice of closed subspace of $\oplus_\theta \mathcal{H}$, but is not the usual Hilbert lattice of Hilbert Quantum Logic, since $\oplus_\theta \mathcal{H}_\theta$ is not separable, so that in particular the lattice is not complete.

(4) Local observable algebras in massive RQFT.

The algebraic description of (massive) Relativistic Quantum Field Theory (RQFT) is based on the following structure (Haag, 1992): an inclusion preserving

map $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$ assigning to each finite contractible open region (or alternatively open double cone) \mathcal{O} in Minkowski space-time, \mathbf{M}_4 , the abstract C^* -algebra of observables measurable in \mathcal{O} . The C^* -algebra generated by the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset \mathbf{M}_4}$ via inductive limit and norm closure is denoted by \mathcal{A} and is called the algebra of quasi-local observables. Locality holds: if $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated, then $\mathcal{A}(\mathcal{O}_1)$ commute with $\mathcal{A}(\mathcal{O}_2)$ elementwise.

Remark. It would be interesting to translate the causal structure underlying the observable net, due to a universal maximal velocity of propagation of information, i.e. $c \neq \infty$, purely in logical terms, like the non-distributivity of the propositional lattice in quantum systems reflects the limitations imposed by $\hbar \neq 0$. Relevant steps in this direction can be found in Haag (1992) and Mundici (1984).

The elements of the Poincaré group \mathcal{P}_+^\uparrow act as automorphisms on the net preserving the local structure. Among the irreducible representations of \mathcal{A} on a separable Hilbert space in which the Poincaré group is unitarily implemented, there is one, π_0 , called the vacuum representation (for simplicity assumed unique) containing a ray, the vacuum, invariant under the unitary representation of \mathcal{P}_+^\uparrow . In infinite systems, as the one considered in RQFT, it appears in concrete examples that physically one should not consider the set of all the representations, but only a subset of “physically realizable” ones. The properties of RQFT at zero temperature and density are discussed in terms of the net $\{\mathfrak{A}(\mathcal{O}) = \pi_0(\mathcal{A}(\mathcal{O}))\}_{\mathcal{O} \subset \mathbf{M}_4}$. $\mathfrak{A}(\mathcal{O})$ can be identified as the “space of bounded continuous observables in the vacuum representation measurable in \mathcal{O} .” In view of Theorem 3.2, $\mathcal{B}(\mathfrak{A}(\mathcal{O})) \simeq \pi_0(\mathcal{A}(\mathcal{O}))''$ (and are these concrete algebras that appear in the constructive approach to RQFT in low dimensions (Glimm and Jaffe, 1987)); since these algebras are von Neumann algebras, $\mathbb{P}(\mathcal{B}(\mathfrak{A}(\mathcal{O})))$ is a complete lattice. A deep result of RQFT with mass gap is that $\pi_0(\mathcal{A}(\mathcal{O}))''$ for \mathcal{O} a double cone is a type III₁ von Neumann algebra (Fredenhagen, 1985), conjectured on physical grounds to be a factor (Haag, 1992). Hence the associated lattice of propositions is non-atomic, the projectors having Murray-von Neumann dimensions only 0, ∞ . In the Baire approach the σ -normal states are the normal states of $\pi_0(\mathcal{A}(\mathcal{O}))''$, however a factor III₁ does not possess pure normal states. Nevertheless in our approach pure logical states corresponding to maximal knowledge on the proposition lattice of the local system are naturally defined, as they are obtained from lifts of states on $\pi_0(\mathcal{A}(\mathcal{O}))$, which being a C^* -algebra with unity has a separating family of pure states.

5. CONCLUSIONS

Summarizing, in this paper we propose that the lattice of elementary propositions of physical systems is completely encoded in the C^* -algebra \mathfrak{A} of “continuous bounded functions or observables” on a generally “non-commutative phase space X ” in the sense of Non Commutative Geometry. The propositional lattice can be

represented as the σ -complete orthomodular lattice of projectors of the space of “(Baire) measurable bounded observables on X ”, which can be obtained as a suitable closure, via the Baire envelope, of \mathfrak{A} . Hence the propositional logic depends on the physical system, but it captures only a very “coarse grained” structure of it. For example it is able to identify the classical or quantum nature of the system and it is sensible to the related “completeness” or “incompleteness” through the verification of the validity of the Lindenbaum property (Giuntini, 1987) in the corresponding logic. But it is also able to distinguish more refined features of quantum systems e.g. the presence of a countable from a non-countable set of superselection sectors or the “dimension” in the sense of Murray–von Neumann of the sectors.

ACKNOWLEDGMENTS

We gratefully acknowledge R. Nobili for many illuminating discussions and for a critical reading of a preliminary version of the manuscript. This work is supported in part by the European Community’s Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime.

REFERENCES

- Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics* **37**, 823–843.
- Cavallaro, S., Morchio, G., and Strocchi, F. (1999). A generalization of the Stone-von Neumann Theorem to nonregular representations of the CCR-Algebra. *Letters in Mathematical Physics* **47**, 307–320.
- Cirelli, R. and Gallone, F. (1973). Algebra of observables and quantum logic. *Annales Institute Henri Poincaré* **A19**, 297–331.
- Clifton, R., Bub, J., and Halvorson, H. (2003). Characterizing quantum theory in terms of information-theoretic constraints. *Foundations of Physics* **33**, 1561–1591.
- Connes, A. (1993). Non Commutative Geometry and Physics. IHES/M/93/32.
- Connes, A. (1994). *Non Commutative Geometry*. Academic Press, San Diego.
- Connes, A., Douglas, M., and Schwarz, A. (1998). Non commutative geometry and matrix theory: compactification on tori. *JHEP* 9802:003.
- Dalla Chiara, M. L. and Giuntini R. (2001). Quantum Logic, e-Print Archive: quant-ph/01001028.
- Davies, E. B. (1968). On the Borel structure of C^* -Algebras. *Communications in Mathematical Physics* **8**, 147–163.
- Doplicher, S., Fredenhagen, K., and Roberts, J. E. (1995). The quantum structure of space-time at the Planck scale and quantum fields. *Communications in Mathematical Physics* **172**, 187–220.
- Duvenhage, R. (2002). The Nature of Information in quantum mechanics. *Foundations of Physics* **32**, 1399–1417.
- Fredenhagen, K. (1985). On the modular structure of local algebras of observables. *Communications in Mathematical Physics* **97**, 79–89.
- Froehlich, J., Grandjean, O., and Recknagel, A. (1995). Supersymmetric quantum theory, noncommutative geometry and gravitation, in *Les Houches 1995, Quantum symmetries* 221–385. e-Print Archive: hep-th/9706132.

- Giuntini, R. (1987). Quantum logics and Lindenbaum property. *Studia Logica* **46**, 17–35.
- Glimm, J. and Jaffe, A. (1987). *Quantum Physics. A Functional Integral Point of View*, Springer, Berlin.
- Haag, R. (1992). *Local Quantum Physics: Fields, Particles, Algebras*, Springer, Berlin.
- Halmos, P. (1950). *Measure Theory*, van Nostrand, Princeton.
- Halvorson, H. (2001). On the nature of continuous physical quantities in classical and quantum mechanics. *Journal of Philosophical Logic* **30**, 27–50.
- Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading.
- Mackey, G. W. (1963). *Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York.
- Mundici, D. (1984). Abstract Model theory and nets of C^* -algebras: noncommutative interpolation and preservation properties, in *Lecture Notes in Math.* **1103**, 351–377, Springer.
- Pedersen, G. K. (1979). *C^* -Algebras and their Automorphism Groups*, Academic Press, London.
- Piron, C. (1964). Axiomatique quantique. *Helvetica Physica Acta* **37**, 439–468.
- Piron, C. (1976). *Foundation of Quantum Physics*, W. A. Benjamin, Reading.
- Plymen, R. J. (1968). C^* -Algebras and Mackey's Axioms. *Communications in Mathematical Physics* **8**, 132–146.
- Primas, H. (1983). *Chemistry, Quantum Mechanics and Reductionism*, Springer, Berlin.
- Redei, M. (1998). *Quantum Logic in Algebraic Approach*, Kluwer, Dordrecht.
- Streater, R. F. (2000). Classical and quantum probability. *Journal of Mathematical Physics* **41**, 3556–3603.
- Teller, P. (1995). *An Interpretative Introduction to Quantum Field Theory*, Princeton University Press, Princeton.
- Thirring, W. (1981). *Quantum Mechanics of Atoms and Molecules. A Course in Mathematical Physics* 3, Springer, New York.
- Witten, E. (2001). Overview of K theory applied to strings, e-Print Archive: hep-th/0007175 *International Journal of Modern Physics* **A16**, 693–706.